

Solution of Faddeev integral equations in configuration space using the hyperspherical harmonics expansion method

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Abstract

A method has been developed to solve three-particle Faddeev equations in the configuration space making use of a series expansion in hyperspherical harmonics. The following parameters of the bound state of triton and helium-3 nuclei have been calculated: the binding energies, the weights of symmetric and mixed-symmetry components of the wave function, the magnetic moments, and the charge radii.

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1. Introduction

Three-nucleon problems take a special place in the nonrelativistic scattering theory, being a key to deeper physical understanding of the structure of many-particle nucleon systems, the processes, in which such systems are engaged, and the very nature of strong interaction. Nowadays, there exist well developed powerful methods for the solution of three-particle problems, the most known of which are the Faddeev method [1-4] and the Kohn–Hulthén variational method, the latter using a series expansion in a basis of correlated hyperspherical harmonics [5-7]. Benchmark calculations testify that both approaches – the variational and Faddeev ones – yield very close results, when describing experiments on Nd -scattering [8,9]. The solution of a system of coupled two-dimensional integral equations, which is required in the Faddeev method, is not a very difficult problem for modern computers used for the description of three-nucleon systems. However, this procedure may turn out rather resource-consuming, when changing to systems with a nucleon number of 4 and more [10]. In recent years, analogous problems have been attracting steady attention of theorists (see, e.g., Refs. [11-13] and the references therein). It is worth noting that, in works by Dzhibuti [14-16], a possibility to reduce the multiplicity of integration in the Faddeev scheme by combining the latter with the hyperspherical

functions method [17-19] was indicated. The equivalence of both approaches was demonstrated earlier in Ref. [20]; namely, the use of hyperspherical harmonic basis gave rise to identical forms of final equations both in the Faddeev method and when solving a three-particle Schrödinger equation. The “hybrid” method proposed in Refs. [14-16] was based on the expansion of Faddeev components in the momentum space into a series of three-particle hyperspherical eigenfunctions mutually coupled by the Raynal–Revai unitary transformation. The bound states of triton and ${}^9_\Lambda\text{Be}$ hypernucleus were described [14], and the problem of continuous spectrum [15] (the reaction of double charge exchange of kaons at triton and helium-3 nuclei) was examined. In Refs. [21] and [22], while describing three- and four-particle bound states, the method of hyperspherical harmonic expansion was used to solve the Faddeev–Yakubovsky integral equations in the momentum representation. This work aimed at studying the capabilities of hyperspherical harmonic expansion technique for the solution of three-particle Faddeev equations in the configuration space. When calculating the bound state parameters, we used semirealistic nucleon-nucleon potentials, which have been used for long in similar problems as test ones by a good many researchers.

The structure of the paper is as follows. In Sec. 2, the general formalism used when deriving the basic equations of the method is expounded. In Sec. 3, a special case of the problem – a bound state of ${}^3\text{H}$ and ${}^3\text{He}$ nuclei – is considered. Here, the approximations are substantiated, and the scheme for the calculation of expansion coefficients of the total wave function Ψ is described. In Sec. 4, the calculation results are reported for the main characteristics of three-nucleon nuclei: the binding energy, the charge form factors, the contributions made by the symmetric and mixed-symmetry components of Ψ , the magnetic moments, and the root-mean-square charge radii. At last, in Sec. 5, a short summary of the work is made.

2. Formalism

We proceed from the well-known Faddeev equations [1] written down for a system of three strongly interacting spinless particles with identical masses m . Let particle 1 be scattered by particles 2 and 3, which are in the bound state:

$$\begin{aligned}\Psi^{(1)} &= \Phi + G_0(Z)T_{23}(Z)(\Psi^{(2)} + \Psi^{(3)}), \\ \Psi^{(2)} &= G_0(Z)T_{31}(Z)(\Psi^{(3)} + \Psi^{(1)}), \\ \Psi^{(3)} &= G_0(Z)T_{12}(Z)(\Psi^{(1)} + \Psi^{(2)}),\end{aligned}\tag{1}$$

where $\Psi^{(1,2,3)}$ are the one-particle wave functions, the sum of which is the total wave function of the system,

$$\Psi = \Psi^{(1)} + \Psi^{(2)} + \Psi^{(3)}, \quad (2)$$

Φ is the asymptotic wave function that describes the infinite motion of particle 1 with respect to the coupled 23-pair, $G_0(Z) = (Z - H_0)^{-1}$ is Green's operator, $Z = E \pm i0$, E is the total energy of the system 1+(23), H_0 is the operator of kinetic energy, and T_{ij} are two-particle transition operators connected with the pair potentials V_{ij} ($ij=12, 23, 31$) by means of the relations:

$$T_{ij}(Z) = V_{ij} + V_{ij}G_0(Z)T_{ij}(Z). \quad (3)$$

Substituting Eq. (3) into Eq. (1) and summing up the equations obtained, we arrive at the formulas [23]:

$$\Psi = (1 - G_0(Z)V_{23})\Phi + G_0(Z)U\Psi, \quad U = V_{12} + V_{23} + V_{31}. \quad (4)$$

One can easily show uniqueness of the solution of this equation. If you execute several consecutive iterations for the Ψ , we can see that the k -th iteration, Ψ_k , will not contain terms with disconnected parts except the $(G_0(Z)V_{23})^{k+1}\Phi$ but this term vanishes, in turn, on the $(k+1)$ -th iteration, etc. Thus the whole infinite series for Ψ will contain only terms with compact kernels. Therefore, the Eq. (4) has a unique solution.

Let us rewrite Eq. (4) as follows:

$$\Psi - \Phi = G_0(Z)U(\Psi - \Phi) + G_0(Z)(V_{12} + V_{31})\Phi, \quad (5)$$

and expand the difference $\Psi - \Phi$ into a series of hyperspherical harmonics (see Appendix A):

$$\Psi - \Phi = \sum_{Kn} \psi_{Kn}(\rho) u_{Kn}(\Omega). \quad (6)$$

Substituting series expansion (6) into Eq. (5) and using the conditions of orthonormalization and completeness for the eigenfunctions of the radial part of operator H_0 , as well as the condition of orthonormalization for hyperspherical harmonics, we obtain the following system of one-dimensional integral equations for expansion coefficients (see Appendix B):

$$\psi_{Kn}(\rho) = f_{Kn}(\rho) + \lambda \sum_{K'n'} R_{KK'}^{nn'}(\rho, \bar{\rho}) \psi_{K'n'}(\bar{\rho}), \quad (7)$$

where

$$f_{Kn}(\rho) = \int d\bar{\rho} Q_K(\rho, \bar{\rho}) W_{Kn}(\bar{\rho}), \quad (8)$$

and

$$R_{KK'}^{nn'}(\rho, \bar{\rho}) = \int d\bar{\rho} Q_K(\rho, \bar{\rho}) U_{KK'}^{nn'}(\bar{\rho}) \quad (9)$$

is the integral operator. The constant λ in Eq. (7) includes numerical coefficients and the nucleon mass m . The functions Q , W , and U in the integrands in Eqs. (8) and (9) look like

$$\begin{aligned} Q_K(\rho, \bar{\rho}) = & -(\bar{\rho}^3/\rho^2) \left\{ I_2(\rho \xi_K(\rho)) K_2(\bar{\rho} \xi_K(\rho)) \Theta(\bar{\rho} - \rho) \right. \\ & \left. + I_2(\bar{\rho} \xi_K(\rho)) K_2(\rho \xi_K(\rho)) \Theta(\rho - \bar{\rho}) \right\}, \end{aligned} \quad (10)$$

$$W_{Kn}(\bar{\rho}) = \int d\Omega u_{Kn}^*(\Omega) (V_{12}(\bar{\rho}, \Omega) + V_{31}(\bar{\rho}, \Omega)) \Phi(\bar{\rho}, \Omega), \quad (11)$$

$$U_{KK'}^{nn'}(\bar{\rho}) = \int d\Omega u_{Kn}^*(\Omega) U(\bar{\rho}, \Omega) u_{K'n'}(\Omega), \quad (12)$$

where

$$\xi_K(\rho) = \sqrt{K(K+4)/\rho^2 - 2mE}, \quad E = E_i - \epsilon, \quad (13)$$

the quantities I_2 and K_2 in Eq. (10) are the modified Bessel functions, whereas E_i and ϵ in Eq. (13) are the energies of incident particle and bound state in the three-particle system ($\epsilon > 0$), respectively. Equations (7) have the most general form. They can be solved using standard numerical routines for an arbitrary set of hyperspherical harmonics.

The procedure used when deriving Eqs. (7) can be easily generalized to the model, in which the wave function Ψ depends on the spins and isospins of nucleons. In this case, the difference $\Psi - \Phi$ is expanded into a series of antisymmetric basis states:

$$\Psi - \Phi = \sum_{Kn} \psi_{Kn}(\rho) \Gamma_{Kn}(\Omega; \sigma, \tau). \quad (14)$$

The states Γ_{Kn} are constructed using the hyperspherical functions $u_{Kn}^{[g]}(\Omega)$ characterized by a definite type of permutation symmetry $[g]$ – antisymmetric, $[a]$, symmetric, $[s]$, and mixed-symmetry, $[']$ and $['']$, ones – and the spin-isospin functions $\xi_n^{[\bar{g}]}(\sigma, \tau)$ with a conjugate symmetry $[\bar{g}]$ – $[s]$, $[a]$, $['']$, and $['']$, respectively (see Appendix C). For instance, in the case of bound state of three-nucleon system ($S=1/2$, $T=1/2$), the quantity Γ_{Kn} looks like [19]

$$\Gamma_{Kn} = u_{Kn}^{[a]} \xi^{[s]} - u_{Kn}^{[s]} \xi^{[a]} + u_{Kn}^{[']} \xi^{['']} - u_{Kn}^{['']} \xi^{[']}. \quad (15)$$

The technique used for the derivation of equations for the radial components $\psi_{Kn}(\rho)$ is similar to that presented above. Besides, the following additional orthogonality relation for $\xi_n(\sigma, \tau)$ is used:

$$\sum_{\sigma\tau} \xi^{[g]\dagger} \xi^{[g']} = \delta_{gg'}. \quad (16)$$

In so doing, we have also to take into account, of course, that the pair potential contains the Pauli matrices and the spin-isospin projection operators.

3. Bound states

Let us consider the problem of the triton bound state making allowance for spin-isospin degrees of freedom of its nucleons. The wave function of ${}^3\text{H}$ can be obtained from Eqs. (5) and (14) by putting the initial condition $\Phi = 0$ in them. Besides, it has also to be taken into account in Eq. (13) that $E_i = 0$ and the total energy E now acquires a sense of binding energy for npn -system: $E \equiv -E_3 = -\epsilon$. Series (14) converges rapidly [19]: the dominant contributions to Ψ are given by the partial components $\psi_{K=0,2,4}$ ($n = \{0000\}$), whereas the contribution made by the partial wave with $K=6$ amounts to about 1% of that produced by ψ_0 . Without loss of generality, let us further assume that pair interaction is central-symmetric,

$$V_{ij} = \sum_{st} V^{(2s+1, 2t+1)}(r_{ij}) P_{ij}^{(2s+1, 2t+1)}(\sigma, \tau), \quad (17)$$

where s and t are possible values of total spin and isospin of the i -th and j -th nucleons, and $P_{ij}^{(2s+1, 2t+1)}$ is the operator of projection onto the spin-isospin state with the multiplicity $(2s+1, 2t+1)$.

Confining expansion (7) to the terms with K ranging from 0 to 4 and using the spin-isospin functions to calculate the matrix elements in the resulting system of equations, we can ultimately write down

$$\begin{aligned} [I + H_{00}^{(+)}] \psi_0 &= H_{02}^{(-)} \psi_2 - H_{04}^{(+)} \psi_4, \\ [2I + H_{22}^{(+)} + H_{22}] \psi_2 &= H_{20}^{(-)} \psi_0 + H_{24}^{(-)} \psi_4, \\ [I + H_{44}^{(+)}] \psi_4 &= -H_{40}^{(+)} \psi_0 + H_{42}^{(-)} \psi_2, \end{aligned} \quad (18)$$

where I is the unit operator,

$$H_{KK'}^{(\pm)} = \frac{3\lambda}{2} \int d\bar{\rho} Q_K(\rho, \bar{\rho}) \int d\Omega u_{Kn}^*(\Omega) \left[V^{(3,1)}(\bar{\rho}, \Omega) \pm V^{(1,3)}(\bar{\rho}, \Omega) \right] u_{K'n'}(\Omega), \quad (19)$$

$$H_{22} = \frac{3\lambda}{2} \int d\bar{\rho} Q_2(\rho, \bar{\rho}) \int d\Omega u_{2n}^*(\Omega) \left[V^{(3,3)}(\bar{\rho}, \Omega) + V^{(1,1)}(\bar{\rho}, \Omega) \right] u_{2n'}(\Omega) \quad (20)$$

are integral operators, and $\lambda = 16m/\pi$. The multi-indexes n and n' are $n = n' = \{0000\}$ in (19) and $n = n' = \{1100\}$ in (20).

The following routine was used to solve system (18). Presenting the latter in the matrix form and zeroing the right hand sides in Eqs. (18), it is easy to verify directly that, for the NN -potentials used in this work (see below), only the first of the three resulting homogeneous equations has a nontrivial solution. Therefore, the iterative procedure becomes determined unambiguously. First, as a first approximation, we find ϕ_0 from the equation $\phi_0 = -H_{00}^{(+)}\phi_0$. The nonzero solution of this equation (the eigenfunction ϕ_0) is known [24] to exist, provided that the matrix of integral operator $H_{00}^{(+)}$ is degenerate. By solving the equation $\det H_{00}^{(+)}(E) = 0$, we find the eigenvalue E , the sense of which is the binding energy of triton. Afterwards, the function ψ_0 determined to an accuracy of a constant factor is substituted into the second and third equations of system (18) to find ψ_2 and ψ_4 . The normalization constant is determined from the condition

$$\int d\mathbf{x} d\mathbf{y} |\Psi|^2 = P_S + P_{S'} = 1, \quad (21)$$

where

$$P_S = \int d\rho \rho^5 (\psi_0^2(\rho) + \psi_4^2(\rho)) \quad (22)$$

and

$$P_{S'} = \int d\rho \rho^5 \psi_2^2(\rho) \quad (23)$$

are the weights of symmetric and mixed-symmetry, respectively, components of the triton wave function. The S - and S' -components combine the states with $L = 0$ and $S = 1/2$. The small P -component ($L = 1$; $S = 1/2, 3/2$), the contribution of which to the normalization integral is of the order of 0.1%, was not taken into account.

The routine used to find the wave function and the binding energy, which was described above, can also be applied to the ${}^3\text{He}$ nucleus. In this case, the kernels of integral operators (19) and (20) must be appended with the Coulomb term $2V_C/3$, where V_C is the Coulomb potential. Certainly, the component with isospin $T = 3/2$ is neglected in the ${}^3\text{He}$ wave function determined from Eqs. (18) using this routine. However, as was shown in Ref. [25], its contribution to normalization integral (21) does not exceed $2.5 \cdot 10^{-3} \%$.

4. Calculation results

The system of equations (18) was solved for a triton in the cases, when the Malfliet–Tjon [26],

$$\begin{aligned} V_t(r) &= (1438.72 e^{-3.11r} - 626.885 e^{-1.55r})/r, \\ V_s(r) &= (1438.72 e^{-3.11r} - 513.968 e^{-1.55r})/r, \end{aligned} \quad (24)$$

Volkov [27, 28],

$$\begin{aligned} V_t(r) &= 144.86 e^{-(r/0.82)^2} - 83.34 e^{-(r/1.6)^2}, \\ V_s(r) &= 0.63 V_t(r), \end{aligned} \quad (25)$$

and Eikemeier–Hackenbroich [29],

$$\begin{aligned} V_t(r) &= 600 e^{-5.5r^2} - 70 e^{-0.5r^2} - 27.6 e^{-0.38r^2}, \\ V_s(r) &= 880 e^{-5.4r^2} - 70 e^{-0.64r^2} - 21 e^{-0.48r^2} \end{aligned} \quad (26)$$

potentials are used.

The determined dependencies $\psi_K(\rho)$ were used to calculate the charge form factors of ${}^3\text{H}$ and ${}^3\text{He}$ nuclei [30, 31]

$$\begin{aligned} F_{ch}({}^3\text{H}) &= 2F_{ch}^n F_L + F_{ch}^p F_O, \\ 2F_{ch}({}^3\text{He}) &= 2F_{ch}^p F_L + F_{ch}^n F_O, \end{aligned} \quad (27)$$

where

$$\begin{aligned} F_L &= F_1 - F_2/3, \\ F_O &= F_1 + 2F_2/3. \end{aligned} \quad (28)$$

In turn, the bulk form factors F_1 and F_2 in Eq. (28) are expressed in terms of ψ_K as follows [32]:

$$F_1(q) = 24\sqrt{3} \int \frac{\rho^5 d\rho}{a^2} [\psi_0^2 J_2(a) - 2\sqrt{3}\psi_0\psi_4 J_6(a) + \psi_2^2(J_2(a) + J_6(a))], \quad (29)$$

$$F_2(q) = 72\sqrt{6} \int \frac{\rho^5 d\rho}{a^2} \psi_0\psi_2 J_4(a), \quad (30)$$

where $a = \sqrt{2/3}q\rho$.

Using the known formula for the charge form factor at small q 's,

$$F_{ch}(q) = 1 - q^2 r_{ch}^2/6 + \dots \quad (31)$$

and Eqs. (27)–(30), we can obtain [32] the following expressions in the approximation $F_{ch}^n = 0$:

$$\begin{aligned} r_{ch}^2({}^3\text{H}) &= (r_{ch}^p)^2 + r_+^2, \\ r_{ch}^2({}^3\text{He}) &= (r_{ch}^p)^2 + r_-^2, \end{aligned} \quad (32)$$

where $r_{ch}^p = 0.842$ fm is the proton charge radius [33], and

$$r_{\pm}^2 = \int \rho^7 d\rho (\sqrt{3}\psi_0^2 \pm \frac{\sqrt{6}}{4}\psi_0\psi_2). \quad (33)$$

The magnitudes of ${}^3\text{H}$ and ${}^3\text{He}$ nuclear magnetic moments can be calculated by expressing them in terms of known magnetic moments of proton, μ_p , and neutron, μ_n , and the weights of symmetric, P_S , and mixed-symmetry, $P_{S'}$, components [34]:

$$\mu = \frac{\mu_p + \mu_n}{2}(P_S + P_{S'}) - T_3 \frac{\mu_p - \mu_n}{2}(P_S - \frac{1}{3}P_{S'}), \quad (34)$$

where $T_3 = 1/2$ for ${}^3\text{H}$ and $T_3 = -1/2$ for ${}^3\text{He}$.

In the table 1, the calculated binding energies, the weights of wave function components, the magnetic moments, and the root-mean-square charge radii for ${}^3\text{H}$ and ${}^3\text{He}$ nuclei are quoted.

Table 1. Properties of ${}^3\text{H}$ and ${}^3\text{He}$ nuclei calculated for various models of NN -potential.

	Malfliet– Tjon	Volkov	Eikemeier– Hackenbroich	Argonne v_{18} + Urbana IX, [24]	Experiment
$E({}^3\text{H})$, MeV	7.981	7.665	8.942	8.479	8.482 [35]
$P_S({}^3\text{H}, K=0)$, %	95.91	95.75	97.03		
$P_S({}^3\text{H}, K=4)$, %	2.67	2.54	1.76		
$P_{S'}({}^3\text{H})$, %	1.42	1.71	1.21	1.05	
$\mu({}^3\text{H})$, nucl. magn.	2.748	2.739	2.755		2.979 [36]
$r_{ch}({}^3\text{H})$, fm	1.667	1.692	1.644		1.755 [37]
$E({}^3\text{He})$, MeV	7.241	6.951	8.165	7.750	7.719 [35]
$P_S({}^3\text{He}, K=0)$, %	95.83	95.89	96.91		
$P_S({}^3\text{He}, K=4)$, %	2.64	2.24	1.74		
$P_{S'}({}^3\text{He})$, %	1.53	1.87	1.35	1.24	
$\mu({}^3\text{He})$, nucl. magn.	– 1.865	– 1.855	– 1.871		– 2.127 [36]
$r_{ch}({}^3\text{He})$, fm	1.814	1.777	1.737		1.959 [37]

5. Conclusions

The study of scattering processes in systems composed of three strongly interacting particles has been a subject of enhanced interest of researchers for a long time. However, only in the first half of the 1990s, the methods were developed, which

allowed high-precision calculations of observable quantities in $3N$ -reactions to be carried out. The method of Faddeev equations and the method of hyperspherical harmonics, which belong to the most known and effective approaches in researching $3N$ -systems, are deservedly classed as such. In this work, those two approaches have been combined together; namely, the series expansion in hyperspherical harmonics was used to find the solutions of Faddeev integral equations in the configuration space. The new approach takes advantage of the problem geometry directly, by representing the solution of Faddeev equations as a series in the eigenfunctions of the angular part of six-dimensional Laplace operator (the hyperspherical harmonics). As a result, the problem is reduced to the solution of a system of one-dimensional integral equations valid for an arbitrary potential. From a comparison between the results obtained and high-precision data (see Table 1), it follows that the method proposed allows the basic characteristics of the bound state of ^3H and ^3He nuclei to be described satisfactorily for the approximations and potentials used in this work. An advantage of the method is also the circumstance that, unlike the works by Dzhibuti [14, 15], it does not use the Raynal–Revai transformation for partial components of hyperspherical functions, because it is the total wave function that is expanded into a series of hyperspherical harmonics.

Appendix A: Hyperspherical harmonics

The general relations for three-particle hyperspherical harmonics look like [16]

$$\begin{aligned}
u_K^{l_x l_y L M}(\Omega) &= \sum_{m_x m_y} (l_x l_y m_x m_y | L M) u_K^{l_x l_y m_x m_y}(\Omega), \\
u_K^{l_x l_y m_x m_y}(\Omega) &= N_K^{l_x l_y} (\cos \theta)^{l_x} (\sin \theta)^{l_y} \\
&\quad \times P_q^{l_y+1/2, l_x+1/2}(\cos 2\theta) Y_{l_x m_x}(\hat{\mathbf{x}}) Y_{l_y m_y}(\hat{\mathbf{y}}), \tag{A.1}
\end{aligned}$$

where

$$\begin{aligned}
N_K^{l_x l_y} &= \sqrt{\frac{2q!(K+2)(q+l_x+l_y+1)}{\Gamma(q+l_x+3/2)\Gamma(q+l_y+3/2)}}, \\
q &= \frac{K-l_x-l_y}{2},
\end{aligned}$$

$$P_q^{\alpha, \beta}(z) = 2^{-q} \sum_{p=0}^q \binom{q+\alpha}{p} \binom{q+\beta}{q-p} (z-1)^{q-p} (z+1)^p$$

is the Jacobi polynomial, $\binom{a}{b} = \frac{\Gamma(a+1)}{b! \Gamma(a-b+1)}$, l_x is the pair orbital moment corresponding to the Jacobi coordinate $\mathbf{x} = \sqrt{1/2}(\mathbf{r}_2 - \mathbf{r}_3)$, and l_y is the orbital momentum of the first particle with respect to the pair center of mass corresponding to the Jacobi coordinate $\mathbf{y} = \sqrt{2/3}(\mathbf{r}_1 - (\mathbf{r}_2 + \mathbf{r}_3)/2)$.

The notation $u_{Kn}(\Omega) \equiv u_K^{l_x l_y LM}(\Omega)$ for hyperspherical harmonics includes the following quantities: K is the hypermoment; n is a multisubscript, which includes the orbital moments l_x and l_y , the total orbital moment L of the relative motion of all three particles, and its projection M ; and $\Omega = \{\Theta, \theta_x, \phi_x, \theta_y, \phi_y\}$ is the set of five angles in the six-dimensional space, which determine the orientation of the six-dimensional vector $\rho = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$.

Appendix B: System of integral equations for expansion coefficients

The kinetic energy operator H_0 in the hyperspherical basis (ρ, Ω) looks like

$$H_0 = T_0 - \frac{\hbar^2}{2m\rho^2} \Delta_\Omega, \quad T_0 = -\frac{\hbar^2}{2m\rho^5} \frac{\partial}{\partial \rho} \rho^5 \frac{\partial}{\partial \rho}. \quad (\text{B.1})$$

The eigenfunctions of operator Δ_Ω are hyperspherical harmonics (A.1)

$$\Delta_\Omega u_{Kn}(\Omega) = -K(K+4)u_{Kn}(\Omega). \quad (\text{B.2})$$

The functions $u_{Kn}(\Omega)$ are mutually orthogonal. They are normalized in a standard way:

$$\int d\Omega u_{Kn}^*(\Omega) u_{K'n'}(\Omega) = \delta_{KK'} \delta_{nn'}. \quad (\text{B.3})$$

The eigenfunctions $\omega_q(\rho)$ of operator T_0 in Eq. (B.1) are determined by the equation [38]

$$T_0 \omega_q(\rho) = \frac{q^2}{2m} \omega_q(\rho), \quad \omega_q(\rho) = \frac{\sqrt{q}}{\rho^2} J_2(q\rho). \quad (\text{B.4})$$

They satisfy the orthonormalization,

$$\int_0^\infty d\rho \rho^5 \omega_q^*(\rho) \omega_{q'}(\rho) = \delta(q - q'), \quad (\text{B.5})$$

and completeness,

$$\int_0^\infty dq \omega_q^*(\rho) \omega_q(\rho') = \frac{1}{\rho^5} \delta(\rho - \rho') \quad (\text{B.6})$$

conditions.

Substituting Eq. (6) into Eq. (5), multiplying both sides of the obtained equation by $u_{K'n'}^*$, integrating over Ω -variables, and taking into account Eq. (B.3), we obtain the system of integral equations for $\psi_{K'n'}(\rho)$:

$$\begin{aligned}\psi_{K'n'}(\rho) &= \int d\Omega u_{K'n'}^*(\Omega) G_0(Z) f(\rho, \Omega) \\ &+ \int d\Omega u_{K'n'}^*(\Omega) G_0(Z) (V_{12} + V_{31}) \Phi,\end{aligned}\quad (\text{B.7})$$

$$f(\rho, \Omega) = U \sum_{Kn} \psi_{Kn}(\rho) u_{Kn}(\Omega), \quad (\text{B.8})$$

where U is the sum of three pair potentials (4).

Now, substitute Green's operator

$$G_0(\Omega) = \left[Z - T_0 + \frac{\hbar^2}{2m\rho^2} \Delta_\Omega \right]^{-1} \quad (\text{B.9})$$

into Eq. (B.7) and expand the function $f(\rho, \Omega)$ (see Eq. (B.8)) into a series of hyperspherical harmonics:

$$f(\rho, \Omega) = \sum_{K''n''} f_{K''n''}(\rho) u_{K''n''}(\Omega), \quad (\text{B.10})$$

$$f_{K''n''}(\rho) = \int d\Omega u_{K''n''}^*(\Omega) f(\rho, \Omega). \quad (\text{B.11})$$

Then, taking Eqs. (B.2) and (B.3) into account, the system of equations (B.7) reads

$$\psi_{K'n'}(\rho) = \left[Z - T_0 - \frac{\hbar^2}{2m\rho^2} K'(K' + 4) \right]^{-1} F(\rho), \quad (\text{B.12})$$

$$F(\rho) = \int d\Omega u_{K'n'}^*(\Omega) \left[f(\rho, \Omega) + (V_{12} + V_{31}) \Phi \right]. \quad (\text{B.13})$$

Now, expand the function $F(\rho)$ in a series of complete system of T_0 -operator (B.4) eigenfunctions,

$$F(\rho) = \int_0^\infty dq \omega_q(\rho) F_q, \quad (\text{B.14})$$

$$F_q = \int_0^\infty d\bar{\rho} \bar{\rho}^5 F(\bar{\rho}) \omega_q^*(\bar{\rho}). \quad (\text{B.15})$$

Substitute Eq. (B.14) into Eqs. (B.12) and (B.13), and use relations (B.4)–(B.6) and (B.15). After all those operations have been carried out, the system of equations (B.12) takes its ultimate form,

$$\begin{aligned}\psi_{K'n'}(\rho) &= \frac{\pi m}{\hbar^2 \rho^2} \int_0^\infty d\bar{\rho} \bar{\rho}^3 P_\pm(\rho, \bar{\rho}) \int d\Omega u_{K'n'}^* \\ &\times \left[U \sum_{Kn} \psi_{Kn}(\bar{\rho}) u_{Kn}(\Omega) + (V_{12} + V_{31}) \Phi(\bar{\rho}, \Omega) \right],\end{aligned}\quad (\text{B.16})$$

$$P_{\pm}(\rho, \bar{\rho}) = -\frac{2}{\pi} \int_0^{\infty} dq q \frac{J_2(q\rho)J_2(q\bar{\rho})}{q^2 - k_{K'}^2 \mp i0}, \quad (\text{B.17})$$

where $k_{K'}^2 \equiv k_{K'}^2(\rho) = k_0^2 - K'(K' + 4)/\rho^2$, $k_0^2 = 2mE/\hbar^2$, and E is the total energy of the system. An integral of type (B.17) can be calculated analytically [39],

$$\int dx x \frac{J_{\nu}(ax)J_{\nu}(bx)}{x^2 + c^2} = \begin{cases} I_{\nu}(bc)K_{\nu}(ac), & 0 < b < a, \text{Re } c > 0, \text{Re } \nu > -1; \\ I_{\nu}(ac)K_{\nu}(bc), & 0 < a < b, \text{Re } c > 0, \text{Re } \nu > -1; \\ I_{\nu}(-bc)K_{\nu}(-ac), & 0 < b < a, \text{Re } c < 0, \text{Re } \nu > -1; \\ I_{\nu}(-ac)K_{\nu}(-bc), & 0 < a < b, \text{Re } c < 0, \text{Re } \nu > -1. \end{cases} \quad (\text{B.18})$$

Now, introducing functions (10)–(13) and taking Eqs. (B.17) and (B.18) into account, the system of equations (B.16) can be rewritten in compact form (7).

Appendix C: Spin-isospin functions for ^3H and ^3He nuclei

Omitting brackets in the notation for symmetry states (15), $\xi^{[g]} \equiv \xi^g$, let us present the spin-isospin functions of three-nucleon system at $S=1/2$ and $T=1/2$ as linear combinations of spin, χ , and isospin, ζ , component products [40]:

$$\xi^s = \frac{1}{\sqrt{2}}(\chi'\zeta' + \chi''\zeta''), \quad \xi^a = \frac{1}{\sqrt{2}}(\chi'\zeta'' - \chi''\zeta'), \quad (\text{C.1})$$

$$\xi' = \frac{1}{\sqrt{2}}(\chi'\zeta'' + \chi''\zeta'), \quad \xi'' = \frac{1}{\sqrt{2}}(\chi'\zeta' - \chi''\zeta''). \quad (\text{C.2})$$

Here, ξ^s and ξ^a are the functions completely symmetric and completely anti-symmetric, respectively, with respect to the permutation of any pair of nucleons; they are basis functions for two corresponding one-dimensional representations of permutation group for three nucleons. Besides, ξ' and ξ'' are the basis functions for a two-dimensional irreducible representation of the same group, which are characterized by intermediate (mixed) symmetry.

The spin and isospin wave functions look like

$$\chi' = \sqrt{2/3}T'(\alpha_2\alpha_3\beta_1), \quad \chi'' = \sqrt{2/3}T''(\alpha_2\alpha_3\beta_1); \quad (\text{C.3})$$

$$^3\text{H} : \zeta' = \sqrt{2/3}T'(b_2b_3a_1), \quad \zeta'' = \sqrt{2/3}T''(b_2b_3a_1), \quad (\text{C.4})$$

$$^3\text{He} : \zeta' = \sqrt{2/3}T'(a_2a_3b_1), \quad \zeta'' = \sqrt{2/3}T''(a_2a_3b_1), \quad (\text{C.5})$$

where

$$T' = (\sqrt{3}/2)[(13) - (12)], \quad T'' = -(23) + [(13) + (12)]/2 \quad (\text{C.6})$$

are permutation operators (the notation (ij) stands for the permutation of the i -th and j -th nucleons). The one-particle spin (isospin) wave functions α_j and β_j (a_j and b_j) correspond to the positive and negative, respectively, spin (isospin) projection of the j -th particle.

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